Test 3

Exc. 1 Show, that given an inner product (f, g) one can generate orthogonal polynomials $\phi_n(x)$ of degree n by a triple-recursion relation, when the two polynomials $\phi_0(x)$ and $\phi_1(x)$ are given

Proof:

1. take $\hat{\phi}_{k+1}(x) = x\,\phi_k(x)$ guess for $(k+1)$th degree polynomial

2. $\phi_{k+1}(x) = \hat{\phi}_{k+1}(x) - \sum_{i=0}^{k} \alpha_i \phi_i(x)$ Gram-Schmidt to obtain orthogonality

with $(\phi_{k+1}, \phi_i) = 0 \quad i = 0, \dots, k$

$\Rightarrow$ gives $\alpha_i \quad i = 0, \dots, k$

$\Rightarrow \phi_{k+1}$

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$$\phi_{k+1}(x) = x\,\phi_k(x) - \sum_{i=0}^{k} \alpha_i \phi_i(x)$$

$$\Rightarrow x\,\phi_k(x) = \phi_{k+1}(x) + \sum_{i=0}^{k} \alpha_i \phi_i(x)$$

$$\Rightarrow [x\phi_0(x) \;\; x\phi_1(x) \;\; \dots \;\; x\phi_k(x)]$$

$$= [\phi_0(x) \;\; \phi_1(x) \;\; \dots \;\; \phi_k(x) \;\; \phi_{k+1}(x)] \underbrace{\begin{bmatrix} * & * & * & - & - & - & * \\ * & * & * & - & - & - & * \\ & \ddots & * & - & - & \dots & * \\ & & \ddots & & & & \vdots \\ & & & \ddots & & & * \\ & & & & \ddots & * & * \\ & & & & & * & * \end{bmatrix}}_{H_{k+2\,k+1}}$$

multiply from left with $\begin{bmatrix} \phi_0(x) \\ \vdots \\ \phi_k(x) \end{bmatrix}$ and integrate all components $\int w(x) \dots \, dx$ to get inner products

$$\Rightarrow \begin{bmatrix} (\phi_0, x\phi_0) & (\phi_0, x\phi_1) & \dots & (\phi_0, x\phi_k) \\ (\phi_1, x\phi_0) & (\phi_1, x\phi_1) & \dots & (\phi_1, x\phi_k) \\ & & & \\ (\phi_k, x\phi_0) & (\phi_k, x\phi_1) & \dots & (\phi_k, x\phi_k) \end{bmatrix} = \begin{bmatrix} (\phi_0, \phi_0) & (\phi_0, \phi_1) & \dots & (\phi_0, \phi_{k+1}) \\ & & & \\ (\phi_k, \phi_0) & (\phi_k, \phi_1) & \dots & (\phi_k, \phi_{k+1}) \end{bmatrix} H$$

$$\Rightarrow \begin{bmatrix} (\phi_0, x\phi_0) & (\phi_0, x\phi_1) & \cdots & (\phi_0, x\phi_k) \\ (\phi_1, x\phi_0) & (\phi_1, x\phi_1) & \cdots & (\phi_1, x\phi_k) \\ & & & \\ (\phi_k, x\phi_0) & (\phi_k, x\phi_1) & \cdots & (\phi_k, x\phi_k) \end{bmatrix} = \begin{bmatrix} (\phi_0, \phi_0) & (\phi_0, \phi_1) & \cdots & (\phi_0, \phi_k) & (\phi_0, \phi_{k+1}) \\ (\phi_1, \phi_0) & (\phi_1, \phi_1) & \cdots & (\phi_1, \phi_k) & (\phi_1, \phi_{k+1}) \\ & & & & \\ (\phi_k, \phi_0) & (\phi_k, \phi_1) & \cdots & (\phi_k, \phi_k) & (\phi_k, \phi_{k+1}) \end{bmatrix} \cdot H_{k+2\,k+1}$$

symmetric

$(\phi_i, \phi_j) = 0 \quad i \neq j$

$$= \begin{bmatrix} (\phi_0, \phi_0) & & & 0 \\ & (\phi_1, \phi_1) & 0 & 0 \\ 0 & & \ddots & \vdots \\ & & (\phi_k, \phi_k) & 0 \end{bmatrix} H_{k+1\,k} = \begin{bmatrix} (\phi_0, \phi_0) & & 0 \\ & (\phi_1, \phi_1) & \\ 0 & & (\phi_k, \phi_k) \end{bmatrix} \cdot H_{k+1\,k+1}$$

needs to be symmetric as well

$\Rightarrow H_{k+1,k+1}$ needs to be tri-diagonal,

Concluding : $H_{k+2\,k+1} = \begin{bmatrix} * & * & & 0 \\ * & \ddots & \ddots & \\ & \ddots & \ddots & * \\ 0 & & * & * \\ & & & * \end{bmatrix}$

$$\Rightarrow \quad \phi_{k+1}(x) = x\,\phi_k - \alpha_k \phi_k(x) - \alpha_{k-1} \phi_{k-1}(x)$$

triple - recursion relation

Exc 2. $f(x)$ on $[-1,1]$

$$f(x) = \begin{cases} 1+x & x \in [-1,0] \\ 1-x & x \in [0,1] \end{cases}$$

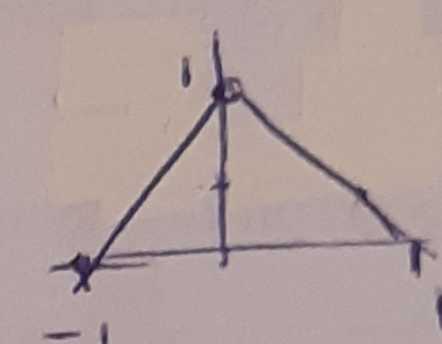
a. $C_n(x) = \sum_{k=0}^{n} a_k T_k(x)$ Chebyshev expansion of $f(x)$, give a_k and show $a_k = 0$ k odd

- $a_k = \dfrac{(f(x), T_k(x))}{(T_k(x), T_k(x))}$

note : • inner product $(f,g) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) g(x)\, dx$

• $(T_i(x), T_j(x)) = 0$ $i \neq j$; orthogonal

- f even function



T_k even for k even
T_k odd for k odd

$\Rightarrow$ for k odd: $(f(x), T_k(x)) = \int_{-1}^{1} \underbrace{\frac{1}{\sqrt{1-x^2}} f(x)}_{\text{even (even . even)}} T_k(x)\, dx = 0$ since integrand odd for k odd and integration from -1 to 1

$\Rightarrow a_k = 0$ k odd

odd k odd
even k even

b. compute a_0, a_2

$a_0 = \dfrac{(f(x), T_0(x))}{(T_0(x), T_0(x))}$ $T_0(x) = 1$

$(f(x), T_0(x)) = (f(x), 1) = 2\int_0^1 (1-x) \cdot 1 \cdot \frac{1}{\sqrt{1-x^2}}\, dx$

$x = \cos\theta$ $dx = -\sin\theta\, d\theta$ $= 2\int_{\frac{1}{2}\pi}^{0} (1-\cos\theta) \frac{1}{\sin\theta} \cdot -\sin\theta\, d\theta$

$= 2\int_0^{\frac{1}{2}\pi} (1-\cos\theta)\, d\theta = 2\,(\theta - \sin\theta)\Big|_0^{\frac{1}{2}\pi} = \pi - 2$

$(T_0(x), T_0(x)) = (1,1) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = \int_{\pi}^{0} \frac{1}{\sin\theta} \cdot -\sin\theta\, d\theta = \int_0^{\pi} d\theta = \pi$

$x = \cos\theta$

$\Rightarrow a_0 = \dfrac{\pi - 2}{\pi} = 1 - \dfrac{2}{\pi}$

$$a_2 : \frac{(f(x), T_2(x))}{(T_2(x), T_2(x))} \qquad T_2(x) = 2x^2 - 1$$

$$(f(x), T_2(x)) = 2\int_0^1 \frac{1}{\sqrt{1-x^2}} (1-x)(2x^2-1)\,dx$$

$x = \cos\theta$

$dx = -\sin\theta\, d\theta$

$$= 2\int_{\frac{1}{2}\pi}^{0} \frac{1}{\sin\theta} (1-\cos\theta)(2\cos^2\theta - 1) . -\sin\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (1-\cos\theta)(2\cos^2\theta - 1)\, d\theta$$

$$= 2\int_0^{\pi/2} (2\cos^2\theta - 1)\, d\theta - 2\int_0^{\pi/2} (2\cos^3\theta - \cos\theta)\, d\theta$$

$$2\int_0^{\pi/2} (2\cos^2\theta - 1)\, d\theta = 2\int_0^{\pi/2} \cos 2\theta\, d\theta = 2 . \frac{1}{2} \sin 2\theta \Big|_0^{\pi/2} = 0$$

$$2\int_0^{\pi/2} (2\cos^3\theta - \cos\theta)\, d\theta = \int_0^{\pi/2} (4\cos^3\theta - 2\cos\theta)\, d\theta$$

$$= \int_0^{\pi/2} (4\cos^3\theta - 3\cos\theta + \cos\theta)\, d\theta$$

$$= \int_0^{\pi/2} (\cos 3\theta + \cos\theta)\, d\theta = \frac{1}{3}\sin 3\theta + \sin\theta \Big|_0^{\pi/2} = -\frac{1}{3} + 1 = \frac{2}{3}$$

$$\Rightarrow (f(x), T_2(x)) = -\frac{2}{3}$$

$$(T_2(x), T_2(x)) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} (2x^2-1)^2\, dx$$

$$= \int_{\pi}^{0} \frac{1}{\sin\theta} (2\cos^2\theta - 1)^2 . -\sin\theta\, d\theta$$

$$= \int_0^{\pi} (2\cos^2\theta - 1)^2\, d\theta = \int_0^{\pi} (\cos 2\theta)^2\, d\theta$$

$$= \int_0^{\pi} \left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right) d\theta = \frac{1}{2}\theta + \frac{1}{8}\sin 4\theta \Big|_0^{\pi} = \frac{\pi}{2}$$

$$a_2 = \frac{-2/3}{\pi/2} = \frac{-4}{3\pi}$$

$$\Rightarrow C_2(x) = 1 - \frac{2}{\pi} - \frac{4}{3\pi}(2x^2 - 1) = 1 - \frac{2}{3\pi} - \frac{8}{3\pi} x^2$$

Exc 3 $\int_0^\infty w(x)\, f(x)\, dx \qquad w(x) = e^{-2x}$

a. Gauss rule with one interpolation point is $\frac{1}{2} f(\frac{1}{2})$

step 1: compute the orthogonal polynomial of degree 1 w.r.t inner product $(f,g) = \int_0^\infty e^{-2x} f(x)\, g(x)\, dx$

- $\varphi_0(x) = 1$
- $\varphi_1(x) = x - \alpha\, \varphi_0(x)$ s.t $(\varphi_0, \varphi_1) = 0$

$$(\varphi_0, \varphi_1) = (1, x - \alpha \varphi_0) = (1,x) - \alpha(1,1) = 0$$

$$\Rightarrow \alpha = \frac{(1,x)}{(1,1)}$$

$$(1,1) = \int_0^\infty e^{-2x} \cdot 1 \cdot 1 \cdot dx = -\frac{1}{2} e^{-2x} \Big|_0^\infty = \frac{1}{2}$$

$$(1,x) = \int_0^\infty \underbrace{e^{-2x}}_{f'} \cdot \underbrace{1 \cdot x}_{g}\, dx \underset{\text{partial int.}}{=} -\frac{1}{2} x e^{-2x} \Big|_0^\infty - \int_0^\infty -\frac{1}{2} e^{-2x}\, dx$$

$$= 0 + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$\Rightarrow \alpha = \frac{(1,x)}{(1,1)} = \frac{1/4}{1/2} = 1/2 \qquad \Rightarrow \varphi_1(x) = x - \frac{1}{2}$$

step 2: the zero of the polynomial of degree 1 is the interpolation point $\Rightarrow x_0 = 1/2$

step 3: exact integration of the interpolating polynomial

$$P_0(x) = f(x_0) \Rightarrow \int_0^\infty e^{-2x} f(x_0)\, dx = f(x_0) \int_0^\infty e^{-2x}\, dx$$

$$= \frac{1}{2} f(x_0) = \frac{1}{2} f(\tfrac{1}{2})$$

$\Rightarrow$ Gauss rule with one interpolation point is $\frac{1}{2} f(\frac{1}{2})$

$$\int_0^\infty e^{-2x} f(x)\, dx = \frac{1}{2} f(\tfrac{1}{2})$$

b. in general Gauss rule with n interpolation points is exact for polynomials with a maximum degree $2n-1$.

Now: $n=1$ $\Rightarrow$ Gauss rule exact for polynomials of degree $2 \cdot 1 - 1 = 1$ maximal

degree of exactness $= 1$

alternative approach:
apply Gauss rule to polynomials

$f(x) = 1$: $\int_0^\infty e^{-2x} \cdot 1 \, dx = \frac{1}{2}$

$\frac{1}{2} f(\frac{1}{2}) = \frac{1}{2} \cdot 1 = \frac{1}{2}$

$\Rightarrow \int_0^\infty e^{-2x} dx = \frac{1}{2} f(\frac{1}{2})$

exact for polynomials of order 0

$f(x) = x$: $\int_0^\infty e^{-2x} x \, dx = \frac{1}{4}$

$\frac{1}{2} f(\frac{1}{2}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$\Rightarrow \int_0^\infty e^{-2x} x \, dx = \frac{1}{2} f(\frac{1}{2})$

exact for polynomials of order 1

$f(x) = x^2$: $\int_0^\infty e^{-2x} x^2 \, dx \underset{p.i.}{=} -\frac{1}{2} x^2 e^{-2x} \Big|_0^\infty - \int_0^\infty -\frac{1}{2} e^{-2x} \cdot 2x \, dx$

$= 0 + \frac{1}{4} = \frac{1}{4}$

$\frac{1}{2} f(\frac{1}{2}) = \frac{1}{2} (\frac{1}{2})^2 = \frac{1}{8}$

$\Rightarrow \int_0^\infty e^{-2x} x^2 \, dx \neq \frac{1}{2} f(\frac{1}{2})$

not exact for polynomials of order 2

$\Rightarrow$ degree of exactness $= 1$